Dragging, perceiving and measuring: physical practices and theoretical exactness in Cabri-environments

Ferdinando Arzarello

arzarello@dm.unito.it

Dipartimento di Matematica

Università di Torino

Italia

Nec manus nuda, nec intellectus sibi permissus, multum valent; instrumentis et auxiliis res perficitur; quibus opus est, non minus ad intellectum, quam ad manum. Atque ut instrumenta manus motum aut cient aut regunt; ita et instrumenta mentis intellectui aut suggerunt aut cavent.

(Francis Bacon, Novum Organum, Aforismos, I, 2)

The unassisted hand, and the understanding left to itself, possess but little power. Effects are produced by the means of instruments and helps, which the understanding requires no less than the hand. And as instruments either promote or regulate the motion of the hand, so those that are applied to the mind prompt or protect the understanding.

Index

Introduction

The paper[1] affords a crucial issue in the use of Cabri, namely perceptual vs. theoretical aspects. In fact, the relationships between perception and theory prove a key element in all mathematics, particularly in geometry. As C. Laborde underlines (Laborde, 1999), diagrams in 2D geometry play an ambiguous role: one the one hand, they refer to theoretical objects, whereas on the other hand they offer *graphical - spatial* properties which can give rise to a perceptual activity from the individual.

Within such a dialectic, dynamic geometry software can have an important mediating role between the two aspects. The interaction concerns deeply perceptual aspects, which involve not only the objects (e. g. drawings) but also the physical perceptions of students, their motions, gestures, languages… and the artefacts that they use as mediating instruments.

The nature of the relationship between the perceptive and the theoretical level is complex and requires a fresh analysis, which entails different components: didactical, cognitive, epistemological. Perceptual aspects which must be analysed concern many components, i.e. visual phenomena, motion, kinaesthesia, inner time(s); on the other side, the most typical theoretical features are the structured mathematical objects, their invariant properties, conjectures, theorems, proofs. Moreover, geometrical entities and their mutual mathematical relationships (e.g. definitions and theorems) have always been in deep connection with the instruments used by geometers to construct their objects of investigation, typically ruler and compasses in Euclid or new drawing machines in Descartes, etc. In fact, the related practises (Fig.1) have deeply determined the same notion of exactness, namely a concept, whose (implicit or explicit) interpretation has been crucial for defining the status of the geometrical objects and the nature of sentences which concern them.

[1] The results presented in this paper are based on a joint research, partially funded by the Italian Ministry of the University (MURST), which involves many Italian researchers. In particular, Ornella Robutti has made a lot of work with the author and with Federica Olivero: see Arzarello et al. (1998a, 1998b), Arzarello (2000), Olivero & Robutti (1999, 2000, 2001a, 2001b). Also the following people have contributed to the theoretical elaboration as well as to the experimental work: P. Accomazzo, V. Andriano, P. Boero, M. G. Bartolini Bussi, G. Gallino, R. Garuti, M. A. Mariotti, M. Maschietto, D. Paola.

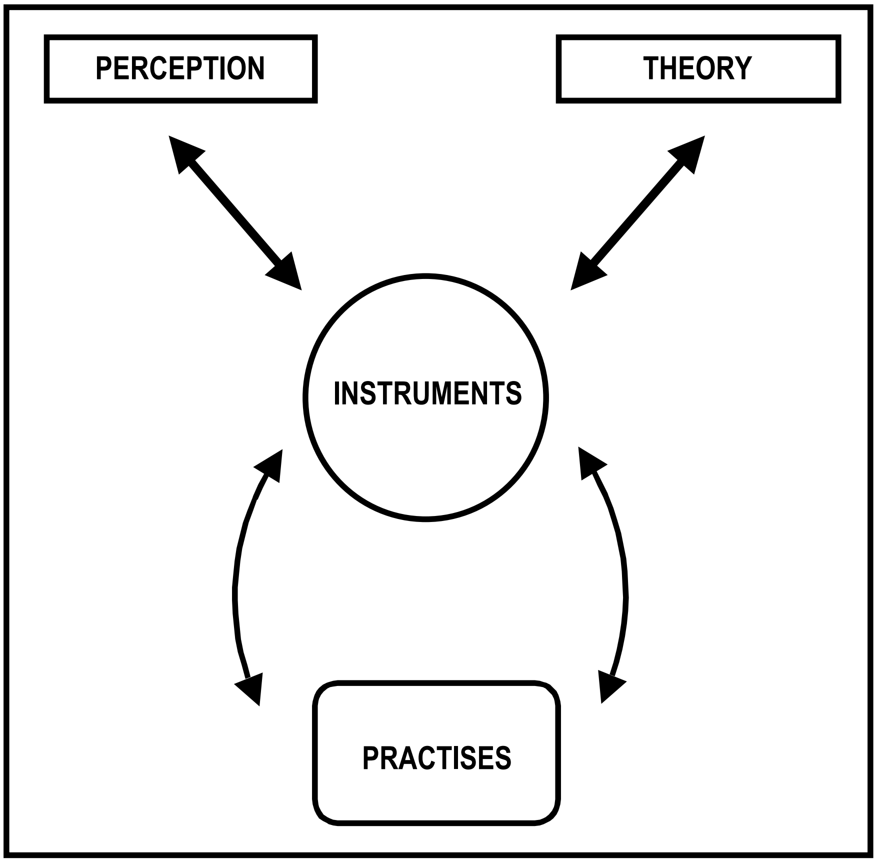

Figure 1

Nowadays such an analysis suggests new insights for looking at the way software for dynamic geometry like Cabri can modify the status of geometrical objects. Precisely, some of its peculiar practises, as dragging and measuring, can mediate the relationship theoretical-perceptual in a specific way, creating entities with a new status. In fact, dragging and measuring are related with important cognitive functions of the subject, namely with the way (s)he looks at and manipulates drawings.

While studying the different ways in which students use measures and dragging in Cabri, our research team has devoted particular attention to the analysis of their language(s), gestures, (inner) times. They assume specific features since subjects interact with artefacts and instruments (e.g. Cabri microworld and dragging), which support a semiotic mediation between perceptions and concepts (because of a specific didactic situation). Learning is so described as a long process of interiorisation, through specific and complex mental dynamics of pupils, from perceptions and actions within technological environments towards structured abstract mathematical objects, embedded in a theoretical framework.

Many theoretical tools and frames are necessary to analyse such a genetic process from a cognitive point of view; specifically, we use:

- the *embodied cognition* perspective (Lakoff & Núñez, 2000);
- the *naturalising phenomenology* approach (Petitot, Varela, Pachoud & Roy, 1999).

With such tools, one can give a fine grain picture of processes in pupils who solve problems within Cabri environments.

1. Perceptual versus theoretical

Roughly speaking, as *perceptual level*, we mean the activities in which the students use perception in a wide sense (e.g. seeing if two points coincide, only by eye); as *theoretical level*, we mean such activities as producing a conjecture in a conditional form and validating it with a proof (e.g. proving that two points coincide).

The relationships between perception and theory prove a key element in mathematics teaching, particularly in geometry, where there is a dialectic between external visual representations, possibly mediated by the technology (information visualisation), and visual imagery (Chiappini & Bottino, 1999). As C. Laborde (1999) points out: "When pupils are asked by the teacher to construct a diagram, the teacher expects them to work at the level of geometry using theoretical knowledge whereas pupils very often stay at the graphical level and try to satisfy only visual constraints."

A didactical consequence of this ambiguity can be put forward analysing such situations in terms of Brousseau's theory of didactic situations (Brousseau, 1998): the graphic 'milieau', namely the system with which the learner is communicating, may be a fleeing object which feeds back to the learner ambiguous answers, because of the fuzzy status of its mathematical objects.

However computers have been analysed as environments which may support a positive dialectic between the two aspects: “the mediation of the computer provides new tools for operating on [abstract objects] and therefore changes the objects themselves” (Laborde, 1999).

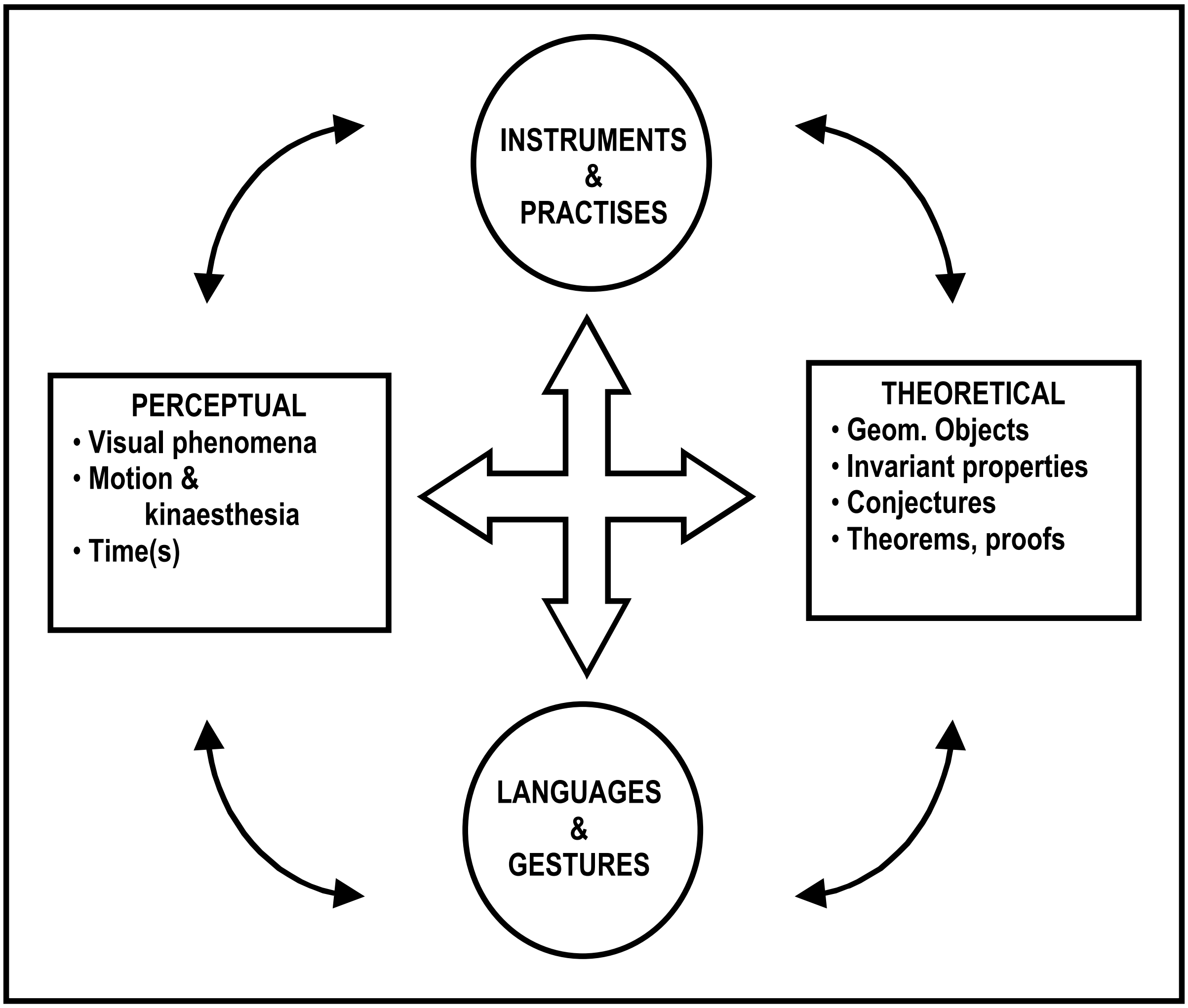

Figure 2

The supported interaction concerns deeply perceptual aspects (see Fig. 2), which involve (Arzarello et al., 1998a, 1998b; Arzarello, 2000):

• the visual representations;

• the perceptions of students, their motions, gestures (kinaesthesia);

• languages (oral, gestual, written,…);

• students’ inner times;

• the artefacts that they use as mediating instruments.

2. The issue of exactness

Within the framework sketched above, where concrete cognitive aspects interact deeply with epistemological aspects, two questions become crucial, which must be answered from a twofold point of view, namely didactical and epistemological (i.e., considering students' minds as well as mathematicians' ideas about that):

- when is a geometrical object *known*?
- when is a problem *solved*?

Historically, such questions have had different answers during the centuries, in deep connection with the instruments used by geometers to construct their objects of investigation. In fact, the instruments have deeply featured the nature of mathematical objects: as a consequence, it was the very notion of *exactness* to change considerably along the time (Bos, 1993). In fact, the *exactness* depends on the technology used by the mathematician to build up a figure, even if the instruments to do that are used in an idealised way.

Let us give some examples. Ruler and compasses have been crucial in the Greek mathematics for defining the nature of the geometrical objects and the related notion of exactness. The issue of exactness breaks down in two crucial historical moments, within classical mathematics, namely with the discovering of *irrational* ratios and with the introduction of so called *mechanical curves*.

In the first case, it was realised that the usual process of calculating a ratio between two segments using a ruler and compasses (i.e., the method of successive divisions by Euclid) could never terminate (like in Fig.3). Hence new methods were necessary to face the problem: to catch exactness ruler and compasses were not enough. It was necessary to develop another kind of *technique*, namely the machinery of proof. (see the whole story in Barbin, 1988).

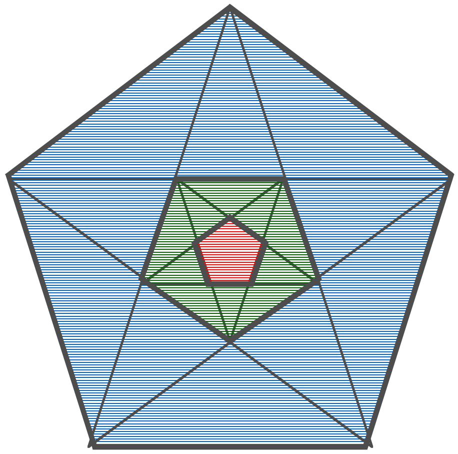

Figure 3 The discovery of irrationals

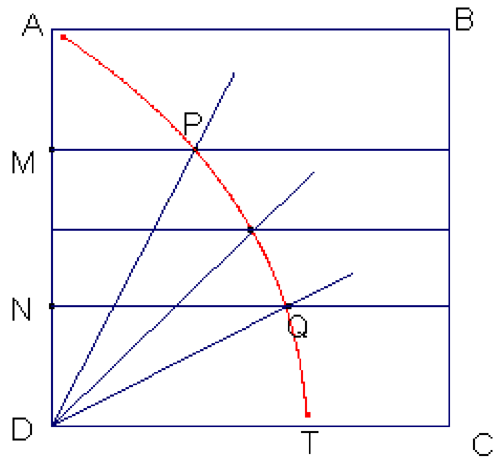

Figure 4 Dinostratus quadratrix

In the second case, it was realised that to solve some 'difficult' problems, e.g. trisecting an angle, ruler and compasses seemed useless while new instruments could do that. For example, one could use the Dinostratus' quadratrix, see Fig.4, namely a curve which is generated by the simultaneous motions of the segment AB, which moves from AB to DC, and by the hand DA, rotating simultaneously around D from position DA to position DC: the intersection points P, Q, … generate a curve which can be used to get the angle QDC, third part of the angle ADC, in correspondence with the segment DN, third part of the segment DA. Exactness here is kept only during the motion of the two segments but it is not possible to stop the instruments when wished in order to mark exactly the point, as with ruler and compasses. The point of intersection is *given* in an essentially different way than working with ruler and compasses. In fact, Clavius modifies the continuous mechanical construction into a discrete *approximation* argument (i. e., by successive bisections of angles and segments, see Bos, 1993).

The new 'compasses' by Descartes (Fig.5) allow to get exactness again for all algebraic curves, as with traditional ruler and compasses.

Ideas on exactness are in continuous evolution even nowadays: a good example is given by the so called *gap theorems* (Hong, 1986, 1987), which are related with mechanical theorem proving, particularly in Geometry and that are a theoretical counterpart to what may be called the pixel syndrome (e.g., the imprecision in figures and measures produced on the computer screen by dynamic geometry software).

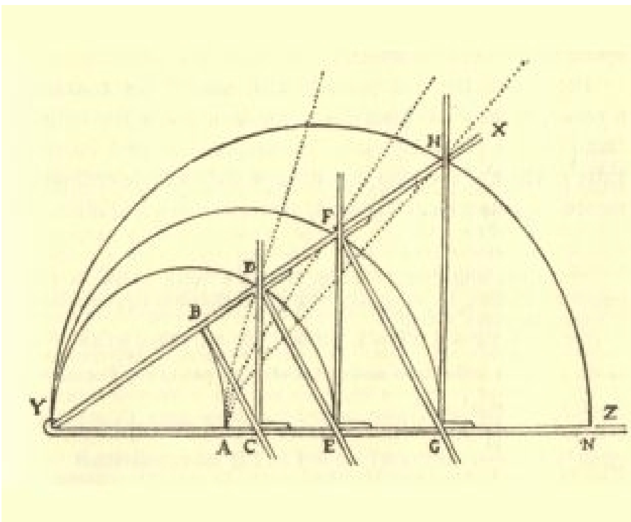

Figure 5 Descartes Compasses

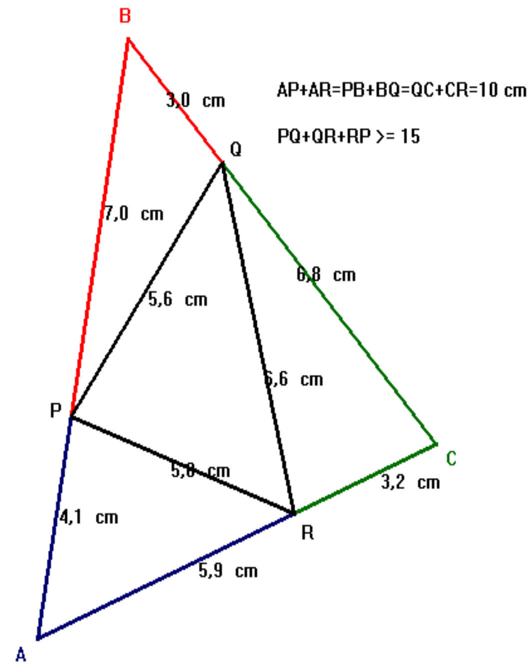

Figure 6

We shall illustrate the flavour of gap theorem with an example: let S be a sentence on a geometric configuration G, which depends on some points A, B, C, …; by the gap theorem, if S is not true, it is so for *many* counter-examples (for *many* points A, B, C,…) and you can verify it with approximations within a fixed value (depending only on the problem). For example (Fig. 6), in a triangle ABC, if the three points P, Q, R are such that AP+AR = BP+BQ = CQ+CR = d, then the perimeter of PQR is not less than 3/2 d. This theorem is difficult to prove using Euclidean geometry. If you explore the situations with Cabri, you realise that you do not find counter-examples; hence it *must* be true. The reasons why things are so, are very deep indeed and are at the roots of the analysis of the complexity of proofs in Geometry, via their translation into a suitable algebraic language (but the gap theorem holds also out of Geometry).

A meaning of the gap theorem is that within specific environments, e.g. in elementary geometry, what we *see* in general is in fact true in general, hence provable from suitable hypotheses (with which we get rid of exceptional cases). Hence our perceptions can give us good hints for conjecturing and proving and consequently such practises as measuring and dragging in Cabri can be a strong tool for building up mathematical (theoretical) knowledge, not only from an practical point of view but also within a theoretical framework.

3. Measuring and dragging

Let us enter into measuring and dragging activities in Cabri with some details. The main points of the analysis can be so summarised:

- such practises as dragging and measuring can mediate the relationship theoretical-perceptual in a specific way, creating entities with a new status (see the SMO in Arzarello, 2000);
- these emerge from problem solving and have institutional and personal features (Mariotti, 2001)

In fact, dragging and measuring support the production of conjectures: exploring figures by moving them, looking at the ways after which their measures, forms, etc. change (or do not change), allows users to discover their invariant properties. The possibility of dragging/measuring offers a feedback to the discovering phase, and in this way it provides support to the role of proofs as real "explanations" of conjectures or properties. These computer-supported practises can be framed within a cognitive evolution back and forth from perceptions to abstract ideas.

In dragging and measuring practises, there are two main cognitive typologies, which can be differently faded according to the concrete situation (Saada-Robert, 1989; Arzarello, 2000; Olivero, 1999):

- *ascending processes*, from drawings to theory, in order to explore freely a situation, looking for regularities, invariants, etc.
- *descending processes* from theory to drawings, in order to validate or refute conjectures, to check properties, etc.

Ascending and descending processes shown by measuring and dragging practises in Cabri reveal cognitive shifts from the perceptual level to the theoretical one and back in students' mathematical activity. Ascending and descending modalities vary during the performance and mark also the way subjects look at what is considered as given and at what is supposed to be found. They constitute a delicate cognitive point, which has also a relevant didactic aspect. It is precisely in these two aspects that one can observe different dynamics between 'pencil & paper' and 'Cabri' environments. In both, the transition is ruled by *abduction*[2]; but while in the former the abductions are produced because of the ingenuity of the subjects, in Cabri the dragging process can mediate them. Moreover, such a repeated switching supports the evolution from perceptions towards a more theoretical frame: this evolution is marked by a kind of rhythm from ascending to descending modalities and back.

Let us illustrate them first in measuring practises (more details in Olivero & Robutti, 2001b). In an ascending modality, measurements are used as a heuristic tool (see the *mesure exploratoire* in Vadcard, 1999) and have a perceptive connotation: the students 'read' measures in order to get ideas about properties, invariants, and relationships of a figure or use measures together with dragging in order to observe how the properties of a figure change, or to discover invariants. In a descending modality, measurements are used as a control tool (see *mesure probatoire* in Vadcard, 1999), in order to check a prediction or to validate a conjecture (e.g. formulated because some regularity has been perceived).

The use of the dragging function in Cabri can be analysed in a similar way: in previous researches we identified different types of dragging which students use according to different purposes during the solution process of open problems and we analysed them modes from a cognitive point of view (Arzarello et al, 1998b; Olivero, 1999). More specifically, observing how students use the mouse while solving a problem in Cabri, F.Olivero discovered the following dragging modalities:

[2] The following example (Peirce, 1960) is illuminating about the meaning of abduction. Suppose I know that a certain bag is plenty of white beans. Consider the following sentences: A) these beans are white; B) the beans of that bag are white; C) these beans are from that bag. A deduction is a concatenation of the form: B and C, hence A; an abduction is: A and B, hence C (Peirce called hypothesis the abduction). An induction would be: A and C, hence B. For more details on abduction see Arzarello et al. (2000).

- Wandering dragging
- Bound dragging
- Guided dragging.
- *Lieu muet* dragging
- Line dragging
- Linked dragging
- Dragging test

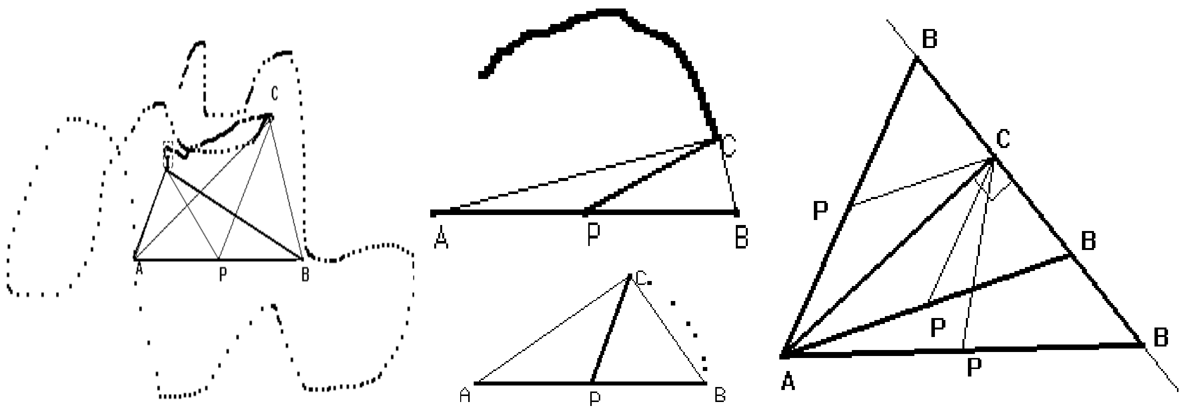

Figure 7 Modalities of dragging: wandering, lieu muet, dragging test

For example (Fig.7), *wandering dragging* means moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the figures; *lieu muet* dragging means moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path (*a dummy locus*), even without being aware of this. As it is well known, a *dragging test* means moving dragable or semi-dragable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the geometric properties you wanted it to have.

Figure 8 illustrates the connections which have been experimentally verified between the ascending/descending modalities and the different typologies of dragging: for example wandering dragging is typical of an ascending modality, while a test dragging is typical of a descending modality.

Of course, the evolution from perceptive to theoretical aspects through ascending/descending modalities does not happen automatically, but a careful didactical design is needed to support students in such a process within dynamic geometry environments (see Mariotti & Bartolini, 1998).

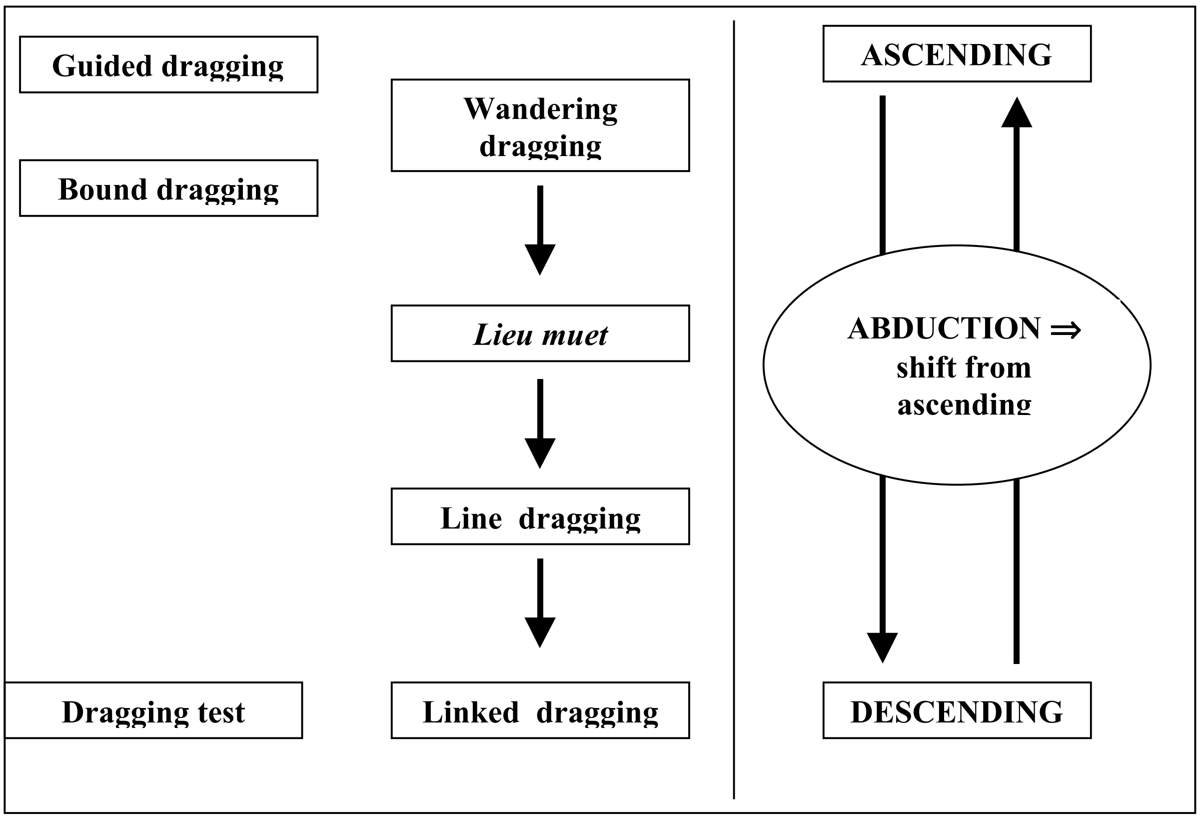

Figure 8

However, things are not always so easy: the basic conflict between theory, figures, proof, from the one side, and perception, drawings, measures, from the other side, are often present in pupils. For example, when students believe that measures are *absolutely exact* or when the dragging exploration is not sufficient to generate productive *abductions*, the shifting from ascending to descending modalities and back may not happen and the evolution towards the theoretical side is blocked.

A major source of conflicts consists in the fact that the Cabri environment *represents* the Euclidean geometry environment from the theoretical point of view, i.e. a mathematical theory in which approximations are not allowed. But it is not always so: in Cabri many things are managed through approximations (measures, check property).

4. Examples of protocols

In this paragraph I'll illustrate conflicts which emerge using Cabri. Sometimes pupils can overcome them, sometimes not. They concern both measuring (ex.1) and dragging (ex. 2) practices.

<u>Example 1.</u> A positive use of measure (Olivero & Robutti, 2000).

> **Problem.** *Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and the intersection points H, L, M, N of pairs of consecutive bisectors.*
>
> *Drag ABCD, considering different configurations. What happens to the quadrilateral HLMN? What kind of figure does it become?*
>
> *Can HLMN become a point? Which hypothesis on ABCD do you need in order to have a point?*
>
> *Write down your conjectures and prove them.*

Here are some excerpts from a peer protocol, followed by some comments [in square brackets]. The two students M, R, 15 y.o., use one computer, one mouse and interact each other; they are novices in the use of Cabri and have been videotaped during their performance (T is the teacher).

……….

90.*T: You need to find a different property from the one you used for the construction.*

91.*R: Let's try to put measures in a particular case.*

92.*M: Draw a square.*

93.They draw a square and then drag one of the sides.

94.*R: One side gets bigger and the other one gets smaller in the same proportion.*

……….

[The students have no idea in mind. They know they have to find a property but they do not know what kind of property it can be. So they decide to start from a particular case. They put measures as a means to get ideas from. They start dragging (#93). While dragging they look at how measures change on the sides of the quadrilateral and compare them. At first a relational information is seen (#94). The figure 9 illustrates their constructions]

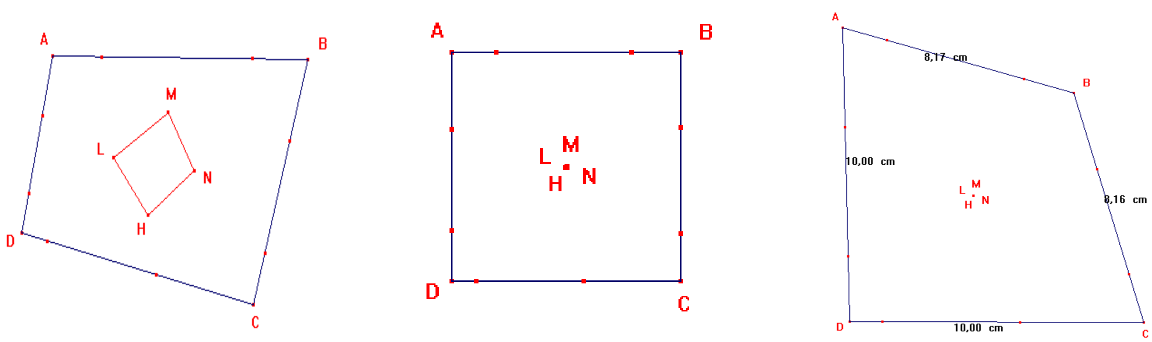

Figure 9

……….

95.*T: How would you say this in mathematics?*

96.M: When one gets bigger and the other one gets smaller, they are inversely proportional.

97.*T: Not really. They are inversely proportional when the product of two variables is constant. Is this the case?*

98.They do some calculations on the measures of the sides of the quadrilateral in different configurations.

[The teacher (T) stimulates the students to move towards the theory *(#95)*. Some work at a theoretical level is done *(#96-97)*. In *#98* they go back again to the Cabri measurements and do some calculations (Fig. 10)]

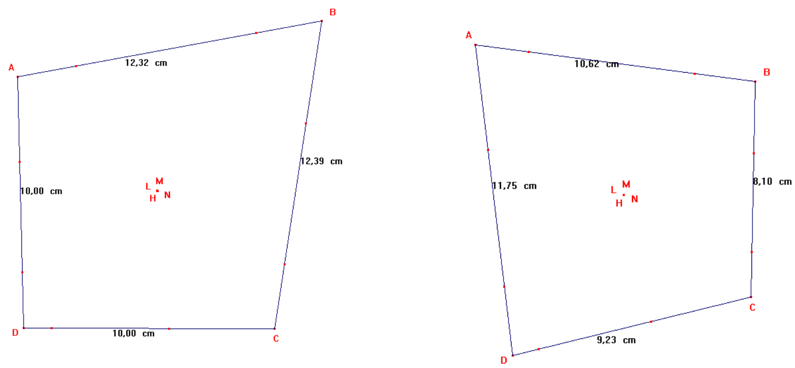

Fig. 10

……….

99.*R: No, the product is not constant.*

100.*M: Let's have a look at the sum.*

101.*R: The sum is constant.*

102.M: So the sum of the two opposite sides is equal. The conjecture is: if a quadrilateral is circumscribed to a circle, the sum of the opposite sides is equal.

……….

[Then they go again to the theory level: relationships between the measurements of the sides are observed and then transformed in a conjecture.(#102)]

However, measures do not always play a positive role in supporting students' conjectures. Sometimes they push students in the opposite direction, because of the approximation used by the software to represent (and give measures of) geometric figures on the screen. Some intriguing example is discussed in Olivero & Robutti (2000, 2001a, 2001b).

<u>Example 2.</u> The story of the *degenerate point:* the role of times and languages (Arzarello, 2000).

Problem. *You are given a quadrilateral ABCD. Construct the perpendicular bisectors of its sides: a, b, c, d. A' is the intersection point of a and b, B' of b and c, C' of c and d, D' of a and d.*

Investigate how A'B'C'D' changes in relation to ABCD.

Prove your conjectures.

Here are some excerpts from another protocol. The three students E, M, V, 17 y.o., use one computer, one mouse and interact each other; they are experts in the use of Cabri and have been videotaped during their performance. A few short comments are inserted in square brackets among the protocol sentences [<u>to distinguish them, characters are underlined</u>], while some more general comments are at the end.

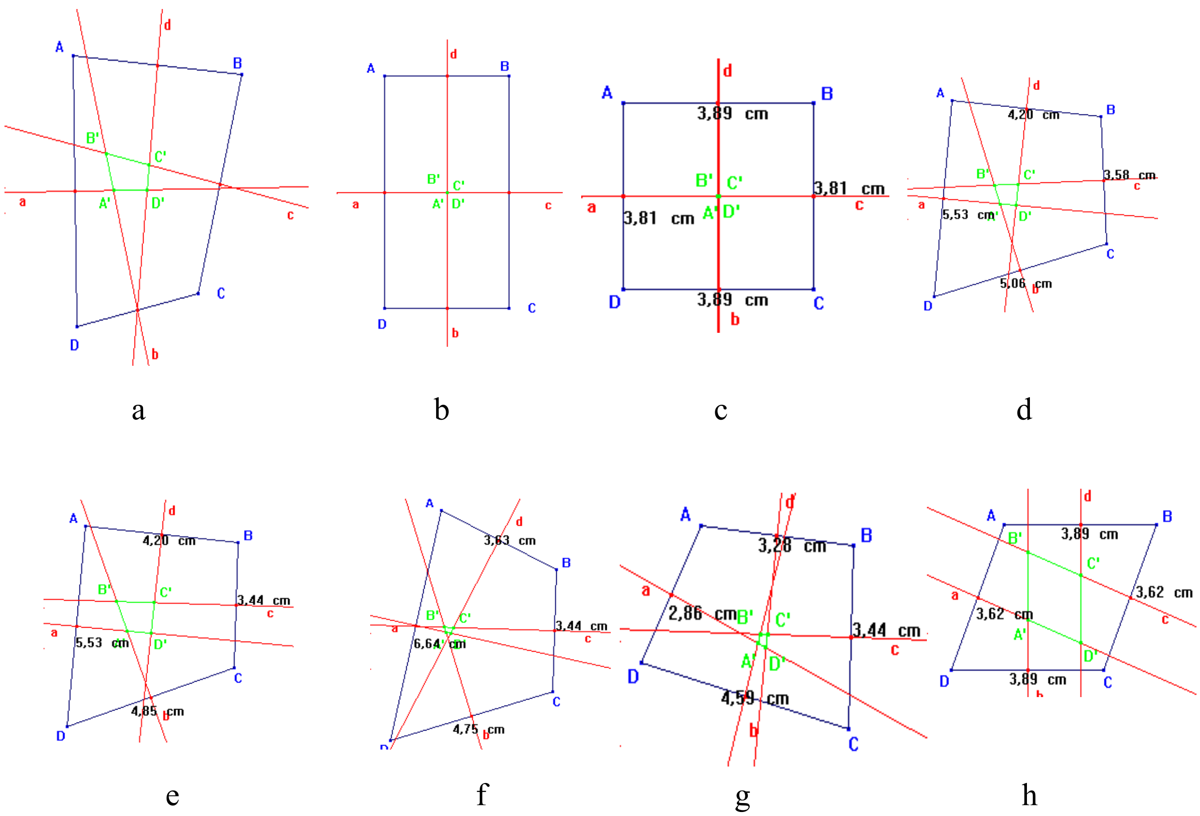

Figure 11

……..

49. *E. And let's try perhaps...let's try to see what happens with regular external quadrilaterals...*

50. *M: I don't know...let's start with a square, so that we see...*

(E drags B, C, D up to get a rectangle, Fig.11b)

[<u>At the beginning, the coincidence of points A', B', C', D' is a purely perceptual fact in an ascending modality (#50)</u>]

...

57. *E: No, I was wondering...that...I was wondering!* (E stops the dragging with Fig.11 c,d,e)

58. *E: No, that is...it degenerates into a point...it's logical isn't it? ...if they are parallel...that is, if the sides are perpendicular...*

[<u>Then it becomes a metaphor which describes the result of a dynamic process (#58)</u>]

59. *M: ...we are looking for...*

60. *E: I mean if the opposite sides are parallel* (she continues dragging B), those [the perp. bisectors] *are perpendicular. And up to this...Isn't it?....and if they are equal the midpoint is on the same line*. (She drags C till ABCD becomes again a rectangle).

61. *M: ok...so?* (E drags A randomly, then D; in the end she goes back to the original figure, Fig. 11, f,g)

62. *E: Please, tell me something!* (E drags B, D, C, A systematically)

63. *V: What are you doing? Are you moving randomly?*

64. *E: No, I was wondering if I could construct a figure...*

65. V: Listen to me, please; let's try thinking...just a moment...'cause of that we have done before...to finish the discourse, when it degenerates into a point, that...have I misunderstood or we have not explained it?

[<u>but its meaning remains to be explained (#65)</u>]

66. *M: well, practically she is saying: since the properties of the perpendicular bisector are perpendicularity and the distance from a point...if...the different segments are parallel, then since they are perpendicular....Moreover if two of...like in a square for example, the midpoint must belong to the same straight line.*

67. *V: yes*

(In the meanwhile E has dragged the points A, B, C, D in order to get a parallelogram, Fig.11h)

68. *E: I am doing a parallelogram...the sides are parallel, aren't they? in the parallelogram. Hence also the perpendicular bisectors are parallel, isn't it? They are parallel two by two.*

[The first chunk of theoretical explanation is obtained through the framing of the perceptual experience in a narrative which encompasses different perceptions]

69. *V&M: yes*

70. *E: So also the segments A'B' and C'D' are parallel.*

71. *V: Hence it maintains...no, nothing!*

(E drags the points B, C, D till she gets a rectangle)

72. *E: hence the square has been proved...degenerate...*

73. V: *Hence if...when...and hmm, yes, that is natural, because wh there are two...the two sides of the external one...the two sides parall two by two, it is natural...that is it should always be that t perpendicular bisectors are...*(Fig.11h)

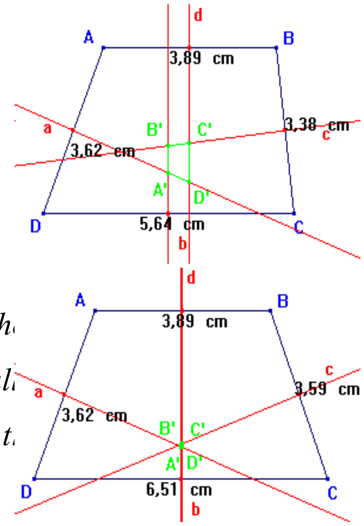

Fig. 12

74. M: *it is so*.

75. *E: Because they are parallel...they are perpendicular to two parallel lines*.

76. *V:...they are parallel...*

77. *E: let's move the point very slowly to see what changes* [she drags the point C for a while]. *Now they are not any longer parallel, hence...these two* [d, b] *are not any longer parallel...sure, it is logic...and not for these two* [a, c] *...That is what we have said up now.*

(E drags slowly point C along line BC and back)

…..

(M takes the mouse and drags B)

92. *E: No! Why the lines...should be so* (she mimics with her hands two parallel horizontal lines); *then it means that one is longer than the other, isn't it?*

94. *E: However if one is longer than the other, the other two are not parallel any longer...Otherwise in the parallelogram they have moved , since they all are parallel.*

(M drags B and stops when the perpendicular bisectors coincide in one point, Fig.12).

95. *V: Hence it can never be..*

96. *E: My God! It degenerates into a point again!* (Fig.12)

[In the end (#96) the word "degenerate" marks the conclusion o a rich exploration framed within a time section in the sense of Boero et al. (1996)]

97. *E: Excuse me, can I...?* (she asks to move the mouse again and drags C)

98. *Let us see what remains equal, when it remains...*

99. *V&M: when?*

100. *E: That is, I try dragging the stuff....and...the point...keeping the point inside...that is moving the point, but leaving that...that...the quadrilateral degenerates into a point.*

[The genesis of the theoretical object is the dragging described in (#100) through the metaphor of "keeping the point inside"]

101. *M: That is, it keeps inside...*

102. *E: ...to find a property that...when it degenerates into a point...do you understand?* (she drags the point B slowly, trying to keep the perpendicular bisectors coincide)

…the students recall what they remember about the remarkable points of a triangle; they are in doubt whether to consider the circumcentre or the centroid; they make some exploration and discuss about the circles outside and inside the quadrilateral...

126. *M: That is when you can put a circle inside.*

127. *E: No, no! I know it! It is the circumcentre..., why it must be equidistant from the sides, isn't it?* (she indicates with fingers on the screen) *This point* (the meeting point of the bisectors) *is the perpendicular bisector of this* (AD) *hence it is equidistant* (from A and D)

[Now things are ready for the final transformation, namely the "degenerate point" to be transformed into the "circumcentre"(#102, 127), see Fig.13]

This protocol is interesting, since it shows students' different types of *languages*, their *inner times* and their mutual *synchronisation* through dragging and measuring practices.

First of all, one can observe students' different inner times, namely:

- the *time of past experience*, that is the lived experiences and memory's action;
- *contemporaneity time*, where perceptions and actions do live;
- the *exploration time*, namely the space of anticipation and volition, towards the future.

The co-ordination of inner times happens through the use of language, that allows to dilate, to contract, to slice duration. Inner time becomes a sort of mental environment where subjects use language as an inner speech, with a non-linear syntax.

For example, the gestures and the broken oral language in #73 are an attempt to grasp the complex relationships among geometrical objects represented by Cabri figures. Such a language is a genuine communication tool (#74). It does not obey the standard rules of (linear) language; it has a specific and multidimensional structure, which I have defined *multivariate* in Arzarello (2000). In fact, it has some connection with the condensed grammar of inner speech, described in Vygotsky (1934).

The transition to the scientific discourse, namely to a linear ordered de-timed language is got through different operations:

- *pruning*
- *the synchronisation of tempos*

The protocol offers an interesting illustration of such cognitive activities; it must be observed that the role played by dragging practises and by the complex interactions among students seems essential for the production of such processes.

Let us start with an example of *pruning*: the multivariate sentence #127 is transformed into the almost linear utterance #131.

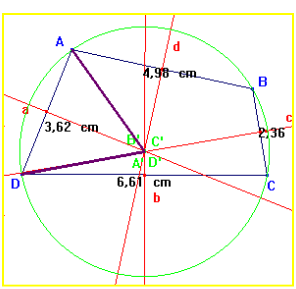

Figure 13

131. E: *...because the bisector is the locus of points which are equi-distant from the extremes...hence it is equidistant from this and from this [A and B]. But from this [A], these two are equal…[she repeats the reasoning and gestures with respect to all vertices] ...hence in the end they are all equal and it is the ray, isn't it?...*

Now let us sketch the *synchronisation of tempos*: it is a regularisation of students rythms (e.g. ascending-descending performances) which is got through:

- a suitable managing of dragging;
- a fastening of the ascending/descending dynamic.

This is shown for example in #77 and #92, where the subject with the mouse slows its movement in order to synchronise her tempo with the rhythm of the other students without the mouse. More precisely, the mouse is in the hand of one subject and the others cannot follow what happens on the screen: the subject tries to overcome the gap between the different tempos, controlling hand gestures and dragging slowly and carefully so that the synchronism between perceptual and cognitive aspects can be established (#77, #102).

These processes produce a converging-integrating process, which entails also the ascending-descending modalities after which the students look at figures. After synchronisation, the subjects can switch together from the perceptual to the theoretical side and back and can go further towards the more abstract mathematical objects. Mental times of students reveal crucial in managing such a generation, as well as the mediation of the artefact, in particular the role of dragging and of the representations of mathematical objects (like Cabri drawings, sketches).

5. Some conclusions and new problems

In this paper we have underlined how the use of a dynamic geometry software can deeply influence the cognitive processes and performances of students, namely their tempos and languages. This seems to depend on such practises as measuring and dragging in dynamic geometry environments. We have also remarked the strong connections existing between the nature of mathematical objects and the practises used by geometers to build them during the centuries.

The main ideas and problems are here summarised in some headlines: each represents a subject worth to be investigated within the frame sketched above. Particularly a precise didactical elaboration is needed (e.g. the role of the teacher, the way to use software in the class, etc.), so that such observed facts can enter into everyday didactical engineering.

1. *The way pupils look at figures changes over time*.

The different directions through which pupils look at the objects are deeply intermingled with their mental times, e.g.:

- order from past to present or from future to present, etc.;
- 'tempos' depending on the ways objects are looked at,
- the dialectic perceptions-theory through dragging and measures;
- the outside-inside dynamics and back.

2. *The genesis of conditionals*

'Tempos', orders, causal and conditional dependencies find a crucial connection in problem solving activities and explorations in dynamic geometry environments: in particular the different orders that subjects attach to experienced facts may change or not in the transition from exploring to conjecturing and to proving (see Boero & Guala, 1999).

3. *Perceptions, times and theories*

During the exploration phase the theory possibly crops up from the past. Together with the perceived object it produces a de-timed, scientific sentence.

4. *The genesis of structured mathematical objects*

The figures once perceived as drawings, are manipulated through dragging and measured: their meaning is framed into conditional sentences, that is the perceptual facts and the generalising functions interact to generate a mathematical sense for the original perceptions.

5. *Practises, cognitive pivots and stumbling blocks*

The dynamic geometric figures and/or the changing measures (attached to them) may become a *cognitive pivot*. The hypotheses incorporated in them may be transformed into new ones through the complex dynamic of action and interpretation within changing tempos and orders of the subjects when interacting.

However, it is not always so: possibly for cognitive reasons, but also because of some problems in the software (e.g., approximations in measure).

In short, what is needed is defining the status of Cabri-objects in a very precise way, both from a cognitive and from an epistemological point of view. At the moment Cabri-objects are still too fuzzy and not so well defined in a global way within a precise mathematical frame.

Consequently, it is not yet clear what one is teaching when using Cabri software, namely which kind of objects and knowledge pupils who are exposed to Cabri practises learn in the end. Particularly, which types of mathematical objects do they own in the end?

In the meanwhile, while deepening the major problems above, I think that a few specific problems must be afforded by people working in didactics and by Cabri- designers.

The former should study how to manage in the class:

- the interiorisation of dragging and of measuring in an aware way
- the cultural carving of students' perceptions through a cognitive apprenticeship:
- the classroom experience
- the use of the multivariate language.

The latter should try designing software, such that the epistemological status of its objects is defined in a less fleeing way (possibly using such results as the gap theorem). Their inspiring 'motto' could be: "*New compasses! New Exactness! New geometry!*".

References

Arzarello, F., Micheletti, C., Olivero, F., Paola, D. & Robutti, O.: 1998a, 'A model for analysing the transition to formal proof in geometry', *Proceedings of PME XXII*, Stellenbosh, South Africa, **2**, 24-31.

Arzarello, F., Gallino, G., Micheletti, C., Olivero, F., Paola, D. & Robutti, O.: 1998b, 'Dragging in Cabri and modalities of transition from conjectures to proofs in geometry', *Proceedings of PME XXII*, Stellenbosch, South Africa, **2**, 32-39.

Arzarello F.: 2000, 'Inside and Outside: Spaces, Times and Language in Proof Production', in: *Proceedings of PME XXIV*, Hiroshima, Japan, **1**, 23-38.

Arzarello, F.; Andriano, V.; Olivero, F. & Robutti,O: 2000, 'Abduction and conjecturing in mathematics', *Philosophica*, 1998, 1, **61**, 77-94.

Barbin, E.: 1988, 'La démonstration mathématique: significations épistemologiques et questions didactiques', *Bulletin de l'APMEP,* n. 366.

Boero, P.; Garuti, R.; Mariotti, M.A.: 1996, 'Some dynamic mental processes underlying producing and proving conjectures', *Proc. PME XX*, Valencia, **2**,121-128.

Brousseau, G.: 1998, *Théorie des situations didactiques*, Grenoble: La Pensée Sauvage.

Bos, H.J.M.: 1993, 'On the interpretation of exactness', in: Czermak, J. (ed.), *Philosophy of Mathematics: Proceedings of the 15th International Wittgenstein-Symposium, 16-23 August 1992, Kirchberg am Wechsel (Austria)*, vol. I, Vienna: Hölder-Pichler-Tempsky, 23-44.

Chiappini G., Bottino R.M.: 1999, 'Visualisation in teaching-learning mathematics: the role of the computer', *Proceedings of GVE'99 (Computer Graphics and Visualization Education '99)*, Eurographics Workshop, Coimbra, Portugal, 3-5 July 99, 147-152.

Guala, E. & Boero, P.: 1999, 'Time Complexity and Learning', *Annals of the New York Academy of Sciences*, vol. 879, 164-167.

Hong Jiawei: 1986, 'Proving by example and gap theorems', in: *Proceedings of the 27th Annual Symposium on the Foundation of Computer Science*, 107-116.

Laborde, C.: 1999, 'The hidden role of diagrams in pupils' construction of meaning in geometry', in: C. Hoyles, J. Kilpatrick (eds), *Meaning and Communication,* Kluwer (to appear).

Lakoff, G. & Núñez, R.: 2000, *Where Mathematics comes from*, New York: Basic Books.

Mariotti, M.A. & Bartolini Bussi, M.G.: 1998, 'From drawing to construction: teacher's mediation within the Cabri environment', *Proceedings of PME XXII*, Stellenbosch, **3**, 247-254.

Mariotti, M. A.: 2001, 'Influence of technologies advances on students' mathematical learning', in: English L., Bartolini Bussi M. G., Jones G., Lesh R., & Tirosh D. (eds.) *Handbook of International Research in Mathematics Education,* Lawrence Erbaum Associates (to appear).

Olivero, F.: 1999, 'Cabri-Géomètre as a mediator in the process of transition to proofs in open geometric situations', in: W.Maull & J.Sharp (eds), *Proceedings of the 4th International Conference on Technology in Mathematics Teaching,* University of Plymouth, UK.

Olivero, F. & Robutti, O.: 2000, 'An exploratory study of students' measurement activity in a dynamic geometry environment', CERME 2, WG 2, electronic publication, http://www.math.auckland.ac.nz/~thomas/ CERME2home.html.

Olivero, F. & Robutti, O.: 2001a, 'Measuring and proving in geometry problems with Cabri', ICME9, WG 11, (in press).

Olivero, F. & Robutti, O.: 2001b, 'An exploratory study of students' measurement activity in a dynamic geometry environment', *Proceedings of PME XXV*, Utrecht, Holland, **4**, 9-16.

Peirce, C.S.: 1960, *Collected Papers*, II, Elements of Logic, Harvard: Harvard University Press, 372.

Petitot, J., Varela F.J., Pachoud B. & Roy J.M.: 1999, *Naturalizing Phenomenology*, Stanford: Stanford University Press.

Saada-Robert, M.: 1989, 'La microgénèse de la rapresentation d'un problème', *Psychologie Française*, 34, 2/3.

Vadcard, L.: 1999, 'La validation en geometrie au College avec Cabri-Géomètre: mesures exploratoires et mesures probatoires', *Petit x*, 50, 5-21.

Vygotsky L.S.: 1934, *My_lenie i rec* (*Thought and Language*); also in: *Sobranie socinenija* (1982-84); Italian translation from the two editions by L. Mecacci, Bari: Laterza, 1992.